UNIVERSITY OF BRISTOL

May/June 2023

Faculty of Engineering

First Examination for the Degrees of Bachelor and Master of Engineering

EMAT10100
ENGINEERING MATHEMATICS I

TIME ALLOWED:
3 HOURS

Section A contains 12 questions carrying 5 marks each.
Answer all questions from section A

Section B contains 5 questions carrying 20 marks each.
Answer three questions from section B

The maximum mark for this paper is 120 marks.

Other Instructions

1. Calculators must be non-programmable.
2. Any numerical answers only need to be given to 3 decimal places unless otherwise stated.
3. Formulae sheets and normal distribution tables are attached.

TURN OVER ONLY WHEN TOLD TO START WRITING

SECTION A

A1. (a) Write the complex number $Z = -\frac{1}{16\sqrt{2}} + \frac{\sqrt{3}}{16\sqrt{2}}j$ in polar form.

(1 mark)

(b) Hence, or otherwise, find all of possible values of $Z^{1/4}$ [*you may express the answer in* **either** *polar form or as real and imaginary parts*] and plot them on the Argand diagram.

(4 marks)

A2. (a) Find the Cartesian equation for the plane P that passes through the three points whose position vectors are given by

$$\mathbf{a}: \ (1,2,3), \quad \mathbf{b}: \ (1,-1,4) \quad \text{and} \quad \mathbf{c}: \ (-1,2,3).$$

(3 marks)

(b) Find the unique point of intersection between the plane P and the straight line that is written in vector form as

$$(x,y,z) = (0,2,-1) + (1,1,1)\,t,$$

where t is an arbitrary parameter

(2 marks)

A3. Given the matrices

$$A = \begin{pmatrix} 1 & 2 \\ -3 & 5 \end{pmatrix}, \quad \text{and } B = \begin{pmatrix} 3 & -2 \\ 1 & 1 \\ 0 & 6 \end{pmatrix},$$

decide which of the following matrix products are well defined:

$$AB, \quad BA, \quad A^T B, \quad A^{-1} B^T.$$

Compute each product that is well defined.

(5 marks)

A4. (a) For what value of the parameter β does the matrix system of equations

$$\begin{pmatrix} 2 & \beta & 3 \\ -1 & 0 & 1 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

not have a unique solution?

(3 marks)

(b) For that value of β, decide whether the equations are inconsistent (no solution) or degenerate (infinitely many solutions), taking care to show your reasoning.

(2 marks)

A5. Use l'Hôpital's rule to determine the value of the following limits, if they are well-defined:

$$\text{(a):}\ \lim_{x\to 0} \frac{x^2}{\sin(x)}, \qquad \text{(b):}\ \lim_{x\to\pi/2} \frac{\cos(x)}{\pi^2 - 4x^2}, \qquad \text{(c):}\ \lim_{x\to 0} \frac{\cos(x) - 1}{x^3},$$

taking care to show your working.

(5 marks)

A6. Find an expression for the indefinite integral

$$\int \frac{x^3 - 10x + 1}{x^2 - 4x + 3}\, \mathrm{d}x\ .$$

(5 marks)

A7. Use the chain rule for partial differentiation to compute $\dfrac{\partial f}{\partial s}$ at $(s,t) = (1,1)$ where

$$f(x,y,z) = \sin(x)\cos(y)e^z$$

and

$$x(s,t) = 2 - s^2 - t^2, \qquad y(s,t) = s - t, \qquad z(s,t) = s + t.$$

(5 marks)

A8. Suppose learner drivers have a 20% chance of failing their driving test indepedently from others. Consider 30 learner drivers who take their test consecutively.

(a) Use the binomial distribution to find the probability to four decimal places that no more than three of these learner drivers fail their test.

[*Recall that the binomial distribution* $X \sim B(n, p)$ *has probability mass function* $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$.]

(2 marks)

(b) Use the normal distribution with appropriate mean and variance to approximate the probability in part (a), giving your answer to four decimal places.

[*There is no need to make a continuity correction*].

(3 marks)

A9. In a factory, car parts are sent randomly to one of two machines for further processing. Unfortunately, the machines do not operate perfectly and sometimes the processing does not pass quality checks. The table below shows the findings of recent quality checks where for randomly sampled car parts, it was assessed which machine was used in processing and for each machine the percentage of quality checks passed was measured.

	samples	quality checks passed
machine 1	4,000	70%
machine 2	6,000	80%

(a) Use these data to construct a complete contingency table for the events A, that a car part passed the quality checks, and B, that a car part was processed by machine 1, and their complements. Take care to show the calculations you perform to complete the table.

(4 marks)

(b) Suppose a car part is found to not pass the quality check. What is the probability that it was processed by machine 2?

(1 mark)

A10. Find the solution $x(t)$ to the ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{4 - 16x^2}\,, \qquad x(0) = \frac{1}{2}\,.$$

(5 marks)

A11. Consider the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = \sin(2t)\,.$$

Given that the complementary function [*solution to the homogeneous equation*] can be written in the form $A\cos(2t) + B\sin(2t)$ where A and B are arbitrary constants, find the particular integral solution $x(t)$.

(5 marks)

A12. Consider the function

$$f(x) = -x^3 + 1 - \arctan(x). \tag{1}$$

(a) Demonstrate graphically that $f(x)$ has a unique real root.

(2 marks)

(b) Recall that *Newton's method* for approximating the root $f(x) = 0$ can be written iteratively as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Starting with $x_0 = 1$, use Newton's method to approximate the root of $f(x)$, defined by Equation (1) to two decimal places.

(3 marks)

SECTION B

B1. (a) Consider the matrix

$$A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} .$$

(i) Compute the determinant of A and the eigenvalues of A.

(3 marks)

(ii) Describe carefully, with the aid of a suitable graph, the effect of the two-dimensional linear transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \end{pmatrix} .$$

(3 marks)

(iii) Explain how the answers to parts (i) and (ii) are related.

(1 mark)

(b) Consider the matrix

$$B = \begin{pmatrix} 0 & -2 & 2 \\ 5 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix} .$$

(i) Find the characteristic polynomial [*that is, the equation satistfied by the eigenvalues* λ *of* B].

(4 marks)

(ii) Given that $\lambda = 3$ is an eigenvalue, find the other two eigenvalues of B.

(3 marks)

(iii) Find an eigenvector corresponding to the eigenvalue $\lambda = 3$.

(3 marks)

(iv) Show that the vectors

$$\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

satisfy $B\mathbf{v}_2 = -2\mathbf{v}_3$ and $B\mathbf{v}_3 = 2\mathbf{v}_2$.
Hence, or otherwise, carefully describe the effect of the 3D linear transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto B \begin{pmatrix} x \\ y \\ z \end{pmatrix} .$$

(3 marks)

B2. (a) Consider the function

$$f(x) = \sin(x)e^{-x}\ .$$

(i) Compute the Maclaurin series of $f(x)$ up to and including cubic terms.

(5 marks)

(ii) Show that stationary points of f [*that is, points at which* $\frac{\mathrm{d}f}{\mathrm{d}x}(x) = 0$] are given by the condition $\tan(x) = 1$.
Hence or otherwise, find all stationary points of f for $0 < x < 2\pi$. and classify them as being either maxima, minima or inflection points. Find the value of of $f(x)$ at each stationary point.

(5 marks)

(iii) Sketch a graph of $f(x)$ for $0 \leq x < 2\pi$, explaining your reasoning and indicating the values of $f(x)$ and x at any special points.

(3 marks)

(b) Consider the function

$$u(x,t) = \sin(kx - mt) - e^{-\alpha t},$$

where k, m and α are positive constants.

(i) Compute the all first and second partial derivatives of $u(x,t)$ with resepect to x and t, expressing the answers in terms of the constants k, m and α.

(5 marks)

(ii) Find the values of constants a and b such that $u(x,t)$ satisfies the equation

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^2 u}{\partial x^2} + be^{-\alpha t}\ ,$$

expressing your answers in terms of the constants k, m and α.

(2 marks)

B3. The probability density function of a continuous random variable X is given by:

$$f_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \leq x \leq 1 \\ 2 - x & \text{for } 1 < x \leq 2 \\ 0 & \text{for } x > 2 \end{cases}$$

(a) **(i)** Sketch this function and find the median and mode of X, stating the definitions for these measures.

(3 marks)

(ii) Compute the mean of X, stating the definition of the mean of a continuous random variable.

(3 marks)

(iii) Compute the variance of X, stating the definition of the variance of a continuous random variable.

(4 marks)

(b) **(i)** Find the cumulative distribution function of X and sketch it.

(6 marks)

(ii) Use the cumulative distribution function of X to find $P(\frac{1}{2} < X \leq \frac{3}{2})$.

(2 marks)

(c) Assume that the random variable Y is independent from X and follows the same distribution.

Find the mean and variance of $X + Y$, stating the results and definitions you use in your calculations.

(2 marks)

B4. (a) The vertical velocity $v(t)$ of a particle of mass $m > 0$ that is allowed to fall under gravity within a resistive medium is supposed to be governed by the ordinary differential equation

$$m\frac{\mathrm{d}v}{\mathrm{d}t} + cv = -mg, \qquad v(0) = -v_0,$$

where c, and v_0 are positive constants. Solve the differential equation to find a general expression for the velocity $v(t)$, in terms of the constants c, g, m and v_0.

(5 marks)

(b) Henceforth in this question we shall assume that y and t are measured in units such that we can assume

$$c = 1, \quad g = 1, \quad m = 1 \quad \text{and } v_0 = \frac{1}{2}.$$

Sketch a graph of $v(t)$ against t. What is the terminal velocity $\lim_{t\to\infty} v(t)$?

(2 marks)

(c) It is proposed that a better model of the resistance of the medium is a quadratic law so that

$$\frac{\mathrm{d}v}{\mathrm{d}t} + v^2 = -1, \qquad v(0) = -\frac{1}{2}.$$

Find the solution for the velocity $v(t)$ in this case.

Why is this not likely to be a good model of the motion?

(5 marks)

(d) It is now supposed that the medium also has elasticity, and that the displacement of the particle $y(t)$ is governed by a differential equation of the form

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{5}{4}y = -1, \qquad y(0) = 1, \quad v(0) = \frac{\mathrm{d}y}{\mathrm{d}t}(0) = -\frac{1}{2}.$$

Find the solution $y(t)$ for the displacement, and $v(t) = \dfrac{\mathrm{d}y}{\mathrm{d}t}(t)$ for the velocity.

What is the terminal velocity in this case?

(8 marks)

turn over/...

B5. Consider an initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{3x}{t} + \frac{\mathrm{e}^t}{t^3}, \quad x(1) = \mathrm{e}^1 + 3. \tag{2}$$

(a) By direct substitution and evaluation, show that

$$x(t) = \frac{\mathrm{e}^t + 3}{t^3}, \tag{3}$$

solves the problem (2). That is, you should show that the solution satisfies the differential equation as well as the initial condition.

(4 marks)

(b) Use Euler's method with

(i) $h = 0.5$,

(ii) $h = 0.25$,

to approximate the value of the solution $x(t)$ to (2) at $t = 2$.

(iii) Decreasing the step size is a common technique for lowering the error of an approximation, however it can be computationally expensive and will not always work. How else can the error be decreased? List at least one additional method.

(8 marks)

(c) A first-order ordinary differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t)$ is said to be in *equilibrium* when $f(x,t) = 0$.

(i) Using the explicit solution given in Equation (3) and the original ODE in Equation (2), show that at an equilibrium

$$t\mathrm{e}^t - 3(\mathrm{e}^t + 3) = 0. \tag{4}$$

(2 marks)

(ii) Starting at $t_a = 3$ and $t_b = 4$, use the bisection method to approximate the time t at which the system is in equilibrium. Perform your estimation to 1 decimal place of accuracy.

(6 marks)

Differentiation

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
k, constant	0
x^n, any constant n	nx^{n-1}
e^x	e^x
$\ln x = \log_e x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x = \frac{\sin x}{\cos x}$	$\sec^2 x$
$\mathrm{cosec}\, x = \frac{1}{\sin x}$	$-\mathrm{cosec}\, x \cot x$
$\sec x = \frac{1}{\cos x}$	$\sec x \tan x$
$\cot x = \frac{\cos x}{\sin x}$	$-\mathrm{cosec}^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cosh x$	$\sinh x$
$\sinh x$	$\cosh x$
$\tanh x$	$\mathrm{sech}^2 x$
$\mathrm{sech}\, x$	$-\mathrm{sech}\, x \tanh x$
$\mathrm{cosech}\, x$	$-\mathrm{cosech}\, x \coth x$
$\coth x$	$-\mathrm{cosech}^2 x$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$

The linearity rule for differentiation

$$\frac{d}{dx}(au + bv) = a\frac{du}{dx} + b\frac{dv}{dx} \qquad a, b \text{ constant}$$

The product and quotient rules for differentiation

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

The chain rule for differentiation

If $y = y(u)$ where $u = u(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

For example,
if $y = (\cos x)^{-1}$, $\frac{dy}{dx} = -1(\cos x)^{-2}(-\sin x)$

Integration

$f(x)$	$\int f(x)\,dx = F(x) + c$	
k, constant	$kx + c$	
x^n, $(n \neq -1)$	$\frac{x^{n+1}}{n+1} + c$	
$x^{-1} = \frac{1}{x}$	$\begin{cases} \ln x + c & x > 0 \\ \ln(-x) + c & x < 0 \end{cases}$	
e^x	$e^x + c$	
$\cos x$	$\sin x + c$	
$\sin x$	$-\cos x + c$	
$\tan x$	$\ln(\sec x) + c$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
$\sec x$	$\ln(\sec x + \tan x) + c$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
$\mathrm{cosec}\, x$	$\ln(\mathrm{cosec}\, x - \cot x) + c$	$0 < x < \pi$
$\cot x$	$\ln(\sin x) + c$	$0 < x < \pi$
$\cosh x$	$\sinh x + c$	
$\sinh x$	$\cosh x + c$	
$\tanh x$	$\ln \cosh x + c$	
$\coth x$	$\ln \sinh x + c$	$x > 0$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a} + c$	$a > 0$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\frac{x-a}{x+a} + c$	$\lvert x\rvert > a > 0$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\frac{a+x}{a-x} + c$	$\lvert x\rvert < a$
$\frac{1}{\sqrt{x^2+a^2}}$	$\sinh^{-1}\frac{x}{a} + c$	$a > 0$
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\frac{x}{a} + c$	$x \geqslant a > 0$
$\frac{1}{\sqrt{x^2+k}}$	$\ln(x + \sqrt{x^2+k}) + c$	
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a} + c$	$-a \leqslant x \leqslant a$
$f(ax+b)$	$\frac{1}{a}F(ax+b) + c$	$a \neq 0$
e.g. $\cos(2x-3)$	$\frac{1}{2}\sin(2x-3) + c$	

The linearity rule for integration

$$\int (af(x) + bg(x))\,dx = a\int f(x)\,dx + b\int g(x)\,dx, \qquad (a, b \text{ constant})$$

Integration by substitution

$$\int f(u)\frac{du}{dx}dx = \int f(u)du \quad \text{and} \quad \int_a^b f(u)\frac{du}{dx}\,dx = \int_{u(a)}^{u(b)} f(u)du$$

Integration by parts

$$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b \frac{du}{dx}v\,dx$$

Alternative form:

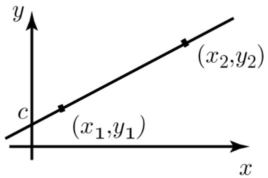
$$\int_a^b f(x)g(x)\,dx = \left[f(x)\int g(x)dx\right]_a^b - \int_a^b \frac{df}{dx}\left\{\int g(x)dx\right\}dx$$

43799 D&PS Jan11

www.mathcentre.ac.uk

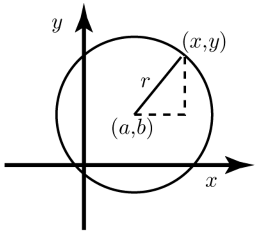
Graphs of common functions

Linear $y = mx + c$, m=gradient, c = vertical intercept



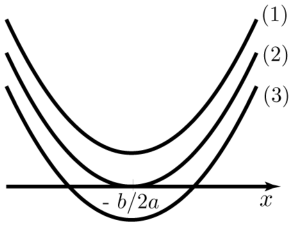
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

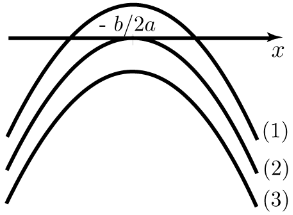
The equation of a circle centre (a, b), radius r



$$(x-a)^2 + (y-b)^2 = r^2$$

Quadratic functions $y = ax^2 + bx + c$





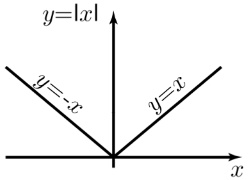
$a > 0$	$a < 0$
(1) $b^2 - 4ac < 0$	(1) $b^2 - 4ac > 0$
(2) $b^2 - 4ac = 0$	(2) $b^2 - 4ac = 0$
(3) $b^2 - 4ac > 0$	(3) $b^2 - 4ac < 0$

Completing the square

$$\text{If } a \neq 0, \quad ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

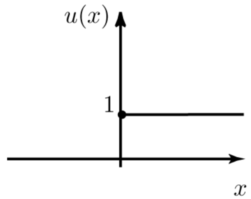
The modulus function

$$|x| = \begin{cases} x & \text{if} \quad x \geqslant 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

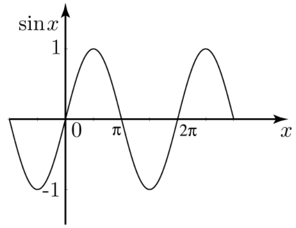


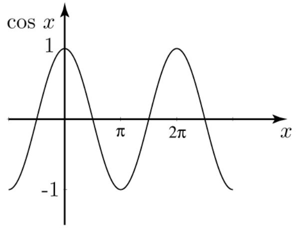
The unit step function, $u(x)$

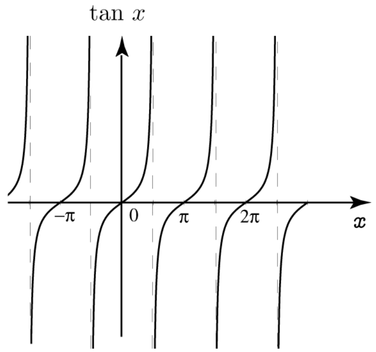
$$u(x) = \begin{cases} 1 & \text{if} \quad x \geqslant 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$



Trigonometric functions

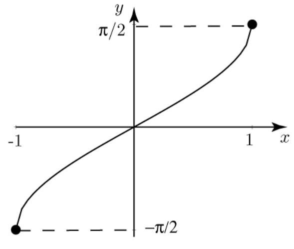




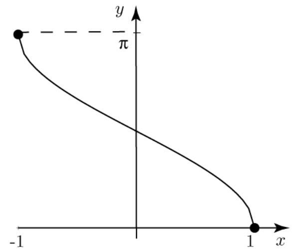


The sine and cosine functions are periodic with period 2π.
The tangent function is periodic with period π.

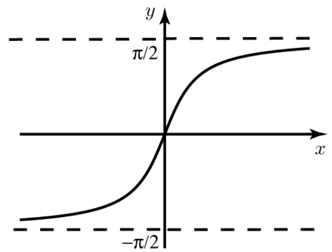
Inverse trigonometric functions



$y = \sin^{-1} x$



$y = \cos^{-1} x$



$y = \tan^{-1} x$

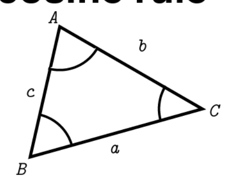
The sine rule and cosine rule

The sine rule

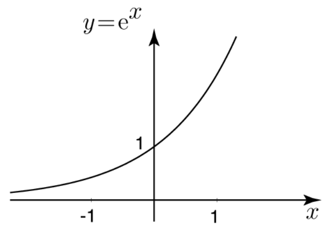
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The cosine rule

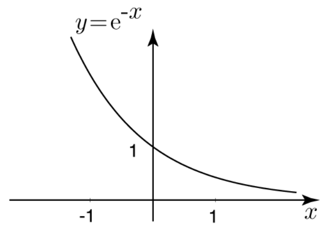
$$a^2 = b^2 + c^2 - 2bc\cos A$$



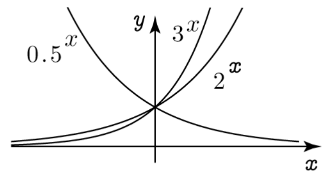
Exponential functions



Graph of $y = e^x$ showing exponential growth

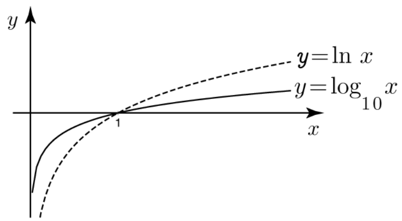


Graph of $y = e^{-x}$ showing exponential decay



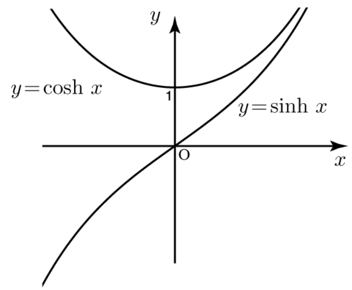
Graphs of $y = 0.5^x$, $y = 3^x$, and $y = 2^x$

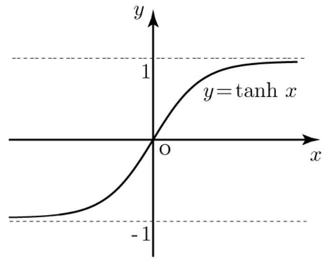
Logarithmic functions



Graphs of $y = \ln x$ and $y = \log_{10} x$

Hyperbolic functions



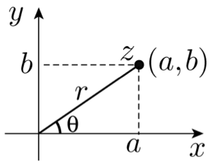


Graphs of $y = \sinh x$, $y = \cosh x$ and $y = \tanh x$

Complex Numbers

Cartesian form: $z = a + bj$
where $j = \sqrt{-1}$

Polar form:
$z = r(\cos\theta + j\sin\theta) = r\angle\theta$
$a = r\cos\theta, \quad b = r\sin\theta,$
$\tan\theta = \frac{b}{a}$



Exponential form: $z = r\mathrm{e}^{j\theta}$

Euler's relations
$\mathrm{e}^{j\theta} = \cos\theta + j\sin\theta, \quad \mathrm{e}^{-j\theta} = \cos\theta - j\sin\theta$

Multiplication and division in polar form

$$z_1 z_2 = r_1 r_2 \angle(\theta_1 + \theta_2), \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)$$

If $z = r\angle\theta$, then $z^n = r^n\angle(n\theta)$

De Moivre's theorem

$$(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta$$

Relationship between hyperbolic and trig functions
$\cos jx = \cosh x, \quad \sin jx = j\sinh x$
$\cosh jx = \cos x, \quad \sinh jx = j\sin x$

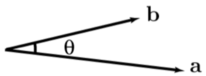
i rather than j may be used to denote $\sqrt{-1}$.

Vectors

If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$

Scalar product

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$

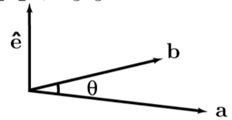


If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Vector product

$\mathbf{a}\times\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\sin\theta\,\hat{\mathbf{e}}$



$\hat{\mathbf{e}}$ is a unit vector perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$ in a sense defined by the right hand screw rule.

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\end{aligned}$$

Sequences and Series

Arithmetic progression: $a, a+d, a+2d, \ldots$
a = first term, d = common difference,
kth term $= a + (k-1)d$
Sum of n terms, $S_n = \frac{n}{2}(2a + (n-1)d)$

Sum of the first n integers,
$1 + 2 + 3 + \ldots + n =$

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$$

Sum of the squares of the first n integers,
$1^2 + 2^2 + 3^2 + \ldots + n^2 =$

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$$

Geometric progression: $a, ar, ar^2, \ldots$
a = first term, r = common ratio,
kth term $= ar^{k-1}$
Sum of n terms, $S_n = \frac{a(1-r^n)}{1-r}$, provided $r \neq 1$

Sum of an infinite geometric series:
$S_\infty = \frac{a}{1-r}, \quad -1 < r < 1$

The binomial theorem
If n is a positive integer

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n$$

When n is negative or fractional, the series is infinite and converges when $-1 < x < 1$

Standard power series expansions

$$\mathrm{e}^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \text{ for all } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \text{ for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \text{ for all } x$$

$$\log_{\mathrm{e}}(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \text{ for } -1 < x \leqslant 1 \text{ only}$$

The exponential function as the limit of a sequence

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x$$

Matrices and Determinants

The 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has determinant

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The 3×3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ has determinant

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(expanded along the first row).

The inverse of a 2×2 matrix
If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
provided that $ad - bc \neq 0$.

Matrix multiplication: for 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \begin{pmatrix} a\alpha + b\beta & a\gamma + b\delta \\ c\alpha + d\beta & c\gamma + d\delta \end{pmatrix}$$

Remember that $AB \neq BA$ except in special cases.

The Binomial Coefficients

The coefficient of x^k in the binomial expansion of $(1+x)^n$ when n is a positive integer is denoted by $\binom{n}{k}$ or ${}^nC_k$.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

$0!=1, \quad n! = n(n-1)!$
so, for example, $4! = 1.2.3.4$

The pattern of the coefficients is seen in

Pascal's triangle:

$$\begin{array}{ccccccccccc} & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

${}^nC_k$ is the number of subsets with k elements that can be chosen from a set with n elements.

Algebra

$$(x+k)(x-k) = x^2 - k^2$$
$$(x+k)^2 = x^2 + 2kx + k^2, \quad (x-k)^2 = x^2 - 2kx + k^2$$
$$x^3 \pm k^3 = (x \pm k)(x^2 \mp kx + k^2)$$

Formula for solving a quadratic equation:

$$\text{if } ax^2 + bx + c = 0 \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Laws of Indices

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (a^m)^n = a^{mn}$$
$$a^0 = 1 \qquad a^{-m} = \frac{1}{a^m} \qquad a^{1/n} = \sqrt[n]{a} \qquad a^{\frac{m}{n}} = (\sqrt[n]{a})^m$$

Laws of Logarithms

For any positive base b (with $b \neq 1$)

$$\log_b A = c \quad \text{means} \quad A = b^c$$
$$\log_b A + \log_b B = \log_b AB, \qquad \log_b A - \log_b B = \log_b \frac{A}{B},$$
$$n \log_b A = \log_b A^n, \qquad \log_b 1 = 0, \qquad \log_b b = 1$$

Formula for change of base: $\log_a x = \frac{\log_b x}{\log_b a}$

Logarithms to base e, denoted $\log_e$ or alternatively ln are called *natural logarithms*. The letter e stands for the exponential constant which is approximately 2.718.

Partial fractions

For *proper fractions* $\frac{P(x)}{Q(x)}$ where P and Q are polynomials with the degree of P less than the degree of Q:

a *linear factor* $ax+b$ in the denominator produces a partial fraction of the form $\frac{A}{ax+b}$

repeated linear factors $(ax+b)^2$ in the denominator produce partial fractions of the form $\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$

a *quadratic factor* ax^2+bx+c in the denominator produces a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$

Improper fractions require an additional term which is a polynomial of degree $n-d$ where n is the degree of the numerator and d is the degree of the denominator.

Inequalities:

$a > b$ means a is greater than b

$a < b$ means a is less than b

$a \geqslant b$ means a is greater than or equal to b

$a \leqslant b$ means a is less than or equal to b

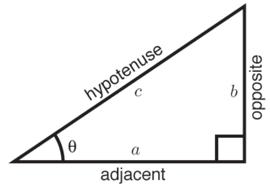
Trigonometry

Degrees and radians

$$360^\circ = 2\pi \text{ radians}, \quad 1^\circ = \frac{2\pi}{360} = \frac{\pi}{180} \text{ radians}$$
$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees} \approx 57.3^\circ$$

Trig ratios for an acute angle θ:

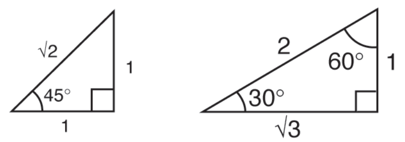
$$\sin\theta = \frac{\text{side opposite to } \theta}{\text{hypotenuse}} = \frac{b}{c}$$
$$\cos\theta = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} = \frac{a}{c}$$
$$\tan\theta = \frac{\text{side opposite to } \theta}{\text{side adjacent to } \theta} = \frac{b}{a}$$



Pythagoras' theorem

$$a^2 + b^2 = c^2$$

Standard triangles:



$$\sin 45^\circ = \frac{1}{\sqrt{2}}, \qquad \cos 45^\circ = \frac{1}{\sqrt{2}}, \qquad \tan 45^\circ = 1$$
$$\sin 30^\circ = \frac{1}{2}, \qquad \cos 30^\circ = \frac{\sqrt{3}}{2}, \qquad \tan 30^\circ = \frac{1}{\sqrt{3}}$$
$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \qquad \cos 60^\circ = \frac{1}{2}, \qquad \tan 60^\circ = \sqrt{3}$$

Common trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

$2 \cos A \cos B = \cos(A-B) + \cos(A+B)$

$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

$\sin^2 A + \cos^2 A = 1$

$1 + \cot^2 A = \mathrm{cosec}^2 A, \quad \tan^2 A + 1 = \sec^2 A$

$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

$\sin 2A = 2 \sin A \cos A$

$\sin^2 A = \frac{1 - \cos 2A}{2}, \qquad \cos^2 A = \frac{1 + \cos 2A}{2}$

$\sin^2 A$ is the notation used for $(\sin A)^2$. Similarly $\cos^2 A$ means $(\cos A)^2$ etc. This notation is used with trigonometric and hyperbolic functions but with positive integer powers only.

Hyperbolic functions

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$
$$\mathrm{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$
$$\mathrm{cosech}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Hyperbolic identities

$e^x = \cosh x + \sinh x, \quad e^{-x} = \cosh x - \sinh x$

$\cosh^2 x - \sinh^2 x = 1$

$1 - \tanh^2 x = \mathrm{sech}^2 x$

$\coth^2 x - 1 = \mathrm{cosech}^2 x$

$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

$\sinh 2x = 2 \sinh x \cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh^2 x = \frac{\cosh 2x + 1}{2}$

$\sinh^2 x = \frac{\cosh 2x - 1}{2}$

Inverse hyperbolic functions

$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \quad \text{for } x \geqslant 1$

$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$

$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \quad \text{for } -1 < x < 1$

The Greek alphabet

A	α	alpha	I	ι	iota	P	ρ	rho
B	β	beta	K	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	T	τ	tau
Δ	δ	delta	M	μ	mu	Υ	υ	upsilon
E	ϵ	epsilon	N	ν	nu	Φ	ϕ	phi
Z	ζ	zeta	Ξ	ξ	xi	X	χ	chi
H	η	eta	O	o	omicron	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega

Written by Tony Croft and Geoff Simpson
for the Mathematics Learning Support Centre
at Loughborough University

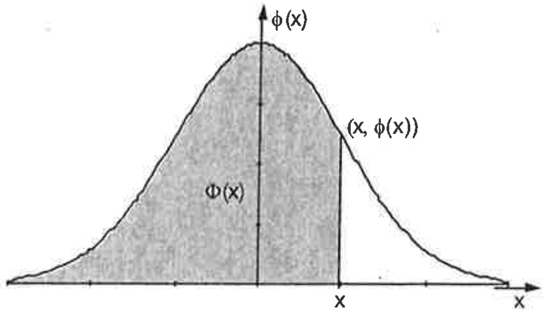
Normal Probability Tables

Table of $\phi(x)$ $[\phi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}]$

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
ϕ(x)	0.3989	0.3970	0.3910	0.3814	0.3683	0.3521	0.3332	0.3123	0.2897	0.2661
x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
ϕ (x)	0.2420	0.2179	0.1942	0.1714	0.1497	0.1295	0.1109	0.0940	0.0790	0.0656
x	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
ϕ (x)	0.0540	0.0440	0.0355	0.0283	0.0224	0.0175	0.0136	0.0104	0.0079	0.0060
X	3.0	3.1	3.2	3.3	3.4	3.5				
ϕ (x)	0.0044	0.0033	0.0024	0.0017	0.0012	0.0009				

Table of $\Phi(x)$ $[\Phi(x)=\int_{-\infty}^{x}\phi(t)dt=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{1}{2}t^2}dt]$

X	0	1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.99903	0.99931	0.99952	0.99966	0.99977	0.99984	0.99989	0.999928	0.999952
4.0	0.999968	0.999979	0.999987	0.9999915	0.9999946	0.9999966	0.9999979	0.9999987	0.99999921	0.99999952



This graph shows $\phi(x)$ -v- x, the shaded area represents $\Phi(x)$